

WEAK ENTROPY SOLUTIONS OF NONLINEAR REACTION-HYPERBOLIC SYSTEMS FOR AXONAL TRANSPORT

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ABSTRACT. This paper is concerned with a class of nonlinear reaction-hyperbolic systems as models for axonal transport in neuroscience. We show the global existence of entropy-satisfying BV-solutions to the initial-value problems by using hyperbolic-type methods. Moreover, we rigorously justify the limit as the biochemical processes are much faster than the transport ones.

Keywords: axonal transport, relaxation limit, difference scheme, BV-estimates, entropy.

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1. INTRODUCTION

The axonal transport is important for the maintenance and functions of nerve cells. These cells are also called *neurons*. A neuron consists of three parts mainly: cell body, dendrites and a single axon. The axon is a long and thin pipe whose length can exceed 10,000 times its diameter. It is this axon that distinguishes neurons from other cells. The axon is responsible for signal transmission in the nervous system. Its cytoplasm does not contain rough endoplasmic reticulum and therefore its proteins can only be transported from the cell body, where all proteins are synthesized.

The transport proceeds as follows. Proteins are stored in vesicles as cargos. The vesicles are attached to kinesin (anterograde motors) or dynein (retrograde motors) proteins. These motor proteins drive the vesicles to walk along the cytoskeletal microtubules as track. Here the kinesin proteins move the vesicles from the cell body to synapse (anterograde transport), while the dynein proteins move the vesicles in the opposite direction (retrograde transport). During the transport, many biochemical processes are possible. For example, the cargos can leave its track, can switch its motor proteins from kinesin to dynein or vice versa, and can move back onto the track. Thus, we can divide the cargos into a number of subpopulations, such as free vesicles, vesicle-kinesin compounds off track, moving vesicle-dynein compounds on track, etc.

As the axon is long and thin, it is reasonable to assume the transport only along the longitudinal direction of the axon. Denote by $x > 0$ the distance down the axon from the cell body which is located at $x = 0$. Let $u_i = u_i(x, t)$ be the concentration at space-time (x, t) of the i -th subpopulations. According to Reed and Blum [6], the mathematical model for axonal transport is partial differential equations of the form:

$$\partial_t u_i + \lambda_i \partial_x u_i = F_i(u_1, u_2, \dots, u_r), \quad i = 1, 2, \dots, r.$$

Here the term $\lambda_i \partial_x u_i$ accounts for the transport of the i -th subpopulation with constant velocity λ_i , and $F_i(u_1, u_2, \dots, u_r)$ describes the biochemical processes of the constituents. It is well recognized that the biochemical processes are much faster than the transport in biosystems.

In [2, 4, 5, 7], the authors studied the linear case, where $F_i(u_1, u_2, \dots, u_r)$ is linear with respect to the u_j 's, in order to explain the approximate traveling waves observed in experiments. Especially, in [5] Friedman and Hu used parabolic-type estimates to analyse the diffusive limit of the linear systems. However, it seems uneasy to deal with the nonlinear problems with the parabolic-type techniques. On the other hand, in [1] Carr showed the existence of global classical solutions to a class of nonlinear models with source terms of the form

$$F_i(u_1, u_2, \dots, u_r) = \begin{cases} f_1, & i = 1 \\ f_i - f_{i-1}, & 1 < i < r, \\ -f_{r-1}, & i = r \end{cases}$$

where $f_i = f_i(u_i, u_{i+1})$ is a continuously differentiable function of two variables. We notice that in applications f_i is a polynomial of u_i and u_{i+1} . In addition, the model in [2] is an example where f_i depends only on u_i and u_{i+1} .

In this paper, we consider the same nonlinear systems, as in [1], but with a small parameter $\epsilon > 0$:

$$(1.1) \quad \begin{cases} (\partial_t + \lambda_1 \partial_x) u_1(x, t) &= \frac{1}{\epsilon} f_1(u_1, u_2), \\ \dots & \\ (\partial_t + \lambda_i \partial_x) u_i(x, t) &= -\frac{1}{\epsilon} f_{i-1}(u_{i-1}, u_i) + \frac{1}{\epsilon} f_i(u_i, u_{i+1}), \\ \dots & \\ (\partial_t + \lambda_r \partial_x) u_r(x, t) &= -\frac{1}{\epsilon} f_{r-1}(u_{r-1}, u_r). \end{cases}$$

Here the small parameter ϵ characterizes the fact that the biochemical processes are much faster than the transport. We assume, throughout this paper, that each f_i is strictly decreasing with respect to the first argument and strictly increasing with respect to the second. In addition, we assume that $f_i(0, 0) = 0$ and, for fixed v , there exists w such that $v f_i(v, w) \geq 0$. These assumptions are consistent with those used in [1, 2, 4, 5, 6, 7].

We will regard (1.1) as a hyperbolic relaxation system [10] and use the corresponding techniques to study it. In particular, we will use a difference scheme to show the global existence of entropy-satisfying BV-solutions $(u_1^\epsilon, u_2^\epsilon, \dots, u_r^\epsilon)$ to the initial-value problems of (1.1) and investigate the limit as ϵ goes to zero. For $r \leq 3$ and for linear problems, we prove that the limit $(u_1^0, u_2^0, \dots, u_r^0)$ is also an entropy-satisfying BV-solution to the so-called equilibrium or reduced system of (1.1):

$$(1.2) \quad \begin{cases} \partial_t(u_1 + u_2 + \dots + u_r) + \partial_x(\lambda_1 u_1 + \lambda_2 u_2 + \dots + \lambda_r u_r) = 0, \\ f_1(u_1, u_2) = f_2(u_2, u_3) = \dots = f_{r-1}(u_{r-1}, u_r) = 0. \end{cases}$$

Although the BV -framework is quite standard for nonlinear hyperbolic problems (see, e.g., [3, 8, 9]), some innovatory ideas seem necessary to carry out the details. In particular, we use the Brouwer fixed-point theorem, the BV -estimate and the special structure of the source terms in (1.1) to derive the existence, boundedness and time-Lipschitz continuity of the difference solutions. It seems not so easy to obtain the time-Lipschitz continuity! Moreover, for an arbitrarily given convex entropy function for the reduced system (1.2), we construct a dissipative entropy function for the original system (1.1). In equilibrium, the constructed entropy function reduces to the given one.

The paper is organized as follows. Section 2 is devoted to the analysis of a difference scheme for the system (1.1). The time-Lipschitz continuity of the difference solutions is derived in Section 3. In Section 4, we discuss entropy functions for the two systems (1.1) and (1.2). The main results are shown in Section 5.

2. DIFFERENCE SOLUTIONS

We begin with construction of approximation solutions to the reaction-hyperbolic system (1.1) by using difference methods. For simplicity, we set

$$U = (u_1, u_2, \dots, u_r)^T, \quad \Lambda = \text{diag}(\lambda_1, \lambda_2, \dots, \lambda_r),$$

$$Q(U) = (f_1, \dots, -f_{i-1} + f_i, \dots, -f_{r-1})^T,$$

where the superscript T denotes the transpose of vectors or matrices. Then the reaction-hyperbolic system (1.1) can be rewritten as

$$(2.1) \quad U_t + \Lambda U_x = \frac{1}{\epsilon} Q(U).$$

About this system, we make the following assumptions mentioned in the introduction:

- (1). each f_i is continuously differentiable, strictly decreasing with respect to the first argument, and strictly increasing with respect to the second;
- (2). $f_i(0, 0) = 0$.

These assumptions are consistent with those used in [1, 2, 4, 5, 6, 7].

Our difference approximation to (2.1) is the following semi-implicit upwind scheme

$$(2.2) \quad \frac{U_j^{n+1} - U_j^n}{\Delta t} + \Lambda^+ \frac{U_j^n - U_{j-1}^n}{\Delta x} + \Lambda^- \frac{U_{j+1}^n - U_j^n}{\Delta x} = \frac{1}{\epsilon} Q_j^{n+1}.$$

Here Δx and Δt denote the increments respectively in x and t ; U_j^n denotes the approximation of $U(x, t)$ over the grid block $[x_j, x_{j+1}) \times [t_n, t_{n+1})$ with $x_j = j\Delta x$ and $t_n = n\Delta t$; $j = 0, \pm 1, \pm 2, \dots$ and $n = 0, 1, 2, \dots$; $Q_j^n = Q(U_j^n)$; $\Lambda^+ = \text{diag}(\lambda_1^+, \lambda_2^+, \dots, \lambda_r^+)$ and $\Lambda^- = \text{diag}(\lambda_1^-, \lambda_2^-, \dots, \lambda_r^-)$ with

$$(2.3) \quad \lambda_i^+ = \frac{\lambda_i + |\lambda_i|}{2} \geq 0 \quad \text{and} \quad \lambda_i^- = \frac{\lambda_i - |\lambda_i|}{2} \leq 0.$$

For $n = 0$, we take

$$(2.4) \quad U_j^0 = \begin{cases} \frac{1}{\Delta x} \int_{x_j}^{x_{j+1}} U_0(x) dx, & \text{if } j \leq \frac{1}{\Delta x} \\ 0, & \text{if } j > \frac{1}{\Delta x} \end{cases},$$

where $U_0(x) := (u_{10}(x), u_{20}(x), \dots, u_{r0}(x))^T$ is a bounded measurable function of $x \in \mathbf{R}$. In addition, the 0 in (2.4) can be replaced with any constant vector.

Throughout this paper, we assume that the grid sizes satisfy the CFL-condition

$$(2.5) \quad \frac{\Delta t}{\Delta x} \max_i |\lambda_i| \leq 1.$$

To analyse the above scheme, we start with the following elementary fact.

Lemma 2.1. *Let $A = (a_{kl})_{n \times n}$ be a matrix satisfying $\sum_{k=1}^n a_{kl} = 1$ for $l = 1, 2, \dots, n$ and $a_{kl} \leq 0$ for $k \neq l$. Then A is invertible, the spectral radius of A^{-1} is not bigger than 1, and the 1-norm of A^{-1} is 1.*

Proof. Let λ be an eigenvalue of A . By the Gershgorin circle theorem there is an integer l such that

$$|\lambda - a_{ll}| \leq \sum_{k \neq l} |a_{kl}| = a_{ll} - 1,$$

for $a_{ll} = 1 - \sum_{k \neq l} a_{kl}$ and $a_{kl} \leq 0$ ($k \neq l$). From this we easily deduce that $|\lambda| \geq 1$. Thus A is invertible and the spectral radius of A^{-1} is not bigger than 1.

Furthermore, it follows from $\sum_{k=1}^n a_{kl} = 1$ that

$$(1, 1, \dots, 1)A = (1, 1, \dots, 1)$$

and thereby

$$(1, 1, \dots, 1)A^{-1} = (1, 1, \dots, 1).$$

This shows that the sum of each column of A^{-1} is also 1. On the other hand, we set $D = \text{diag}(a_{11}, a_{22}, \dots, a_{nn})$ and $F = D - A$. It is obvious that the elements of FD^{-1} are nonnegative,

$$A = D - F = (I - FD^{-1})D$$

and thereby

$$A^{-1} = D^{-1}(I - FD^{-1})^{-1}.$$

Moreover, with the Gershgorin circle theorem it is not difficult to see that the spectral radius of FD^{-1} is less than 1. Thus we can write

$$A^{-1} = D^{-1}(I + FD^{-1} + (FD^{-1})^2 + \dots),$$

which shows that the elements of A^{-1} are nonnegative. Hence the 1-norm of A^{-1} is 1 for the sum of each column of A^{-1} is 1. This completes the proof. \square

Set $V = (v_1, v_2, \dots, v_r)^T$, $W = (w_1, w_2, \dots, w_r)^T$,

$$(2.6) \quad A_i = A_i(V, W) = \int_0^1 \frac{\partial f_i}{\partial u_i} (\theta v_i + (1 - \theta)w_i, \theta v_{i+1} + (1 - \theta)w_{i+1}) d\theta,$$

$$(2.7) \quad B_i = B_i(V, W) = \int_0^1 \frac{\partial f_i}{\partial u_{i+1}} (\theta v_i + (1 - \theta)w_i, \theta v_{i+1} + (1 - \theta)w_{i+1}) d\theta.$$

Assumption (1) implies that $A_i < 0$ and $B_i > 0$. By the mean-value theorem we have

$$(2.8) \quad Q(V) - Q(W) = \mathcal{Q}(V, W)(V - W).$$

From the definition of Q , we have

$$\mathcal{Q}(V, W) = \begin{pmatrix} A_1 & B_1 & & & \\ -A_1 & -B_1 + A_2 & B_2 & & \\ & -A_2 & -B_2 + A_3 & B_3 & \\ & & \cdots & \cdots & \cdots \\ & & & -A_{r-1} & -B_{r-1} \end{pmatrix}.$$

It is easy to verify that $I - \frac{\Delta t}{\epsilon} \mathcal{Q}(V, W)$ satisfies the conditions in Lemma 2.1. Therefore we have

Corollary 2.2. *$I - \frac{\Delta t}{\epsilon} \mathcal{Q}(V, W)$ is invertible, the spectral radius of $(I - \frac{\Delta t}{\epsilon} \mathcal{Q}(V, W))^{-1}$ is not bigger than 1, and the 1-norm of $(I - \frac{\Delta t}{\epsilon} \mathcal{Q}(V, W))^{-1}$ is 1 for any $V, W \in \mathbb{R}^r$.*

With these preparations, we study the difference scheme (2.2). In what follows, we will denote by $|\cdot|$ the 1-norm of vectors and matrices.

Lemma 2.3. *Given U_j^n, U_j^{n+1} can be uniquely determined by solving the nonlinear algebraic equation (2.2).*

Proof. From $f_i(0, 0) = 0$ we see that $Q(0) = 0$ and

$$Q_j^{n+1} = Q_j^{n+1} - Q(0) = \mathcal{Q}(U_j^{n+1}, 0)U_j^{n+1}.$$

Thus, we deduce from the difference scheme (2.2) that

$$(I - \frac{\Delta t}{\epsilon} \mathcal{Q}(U_j^{n+1}, 0))U_j^{n+1} = \frac{\Delta t}{\Delta x} \Lambda^+ U_{j-1}^n + (I - \frac{\Delta t}{\Delta x} \Lambda^+ + \frac{\Delta t}{\Delta x} \Lambda^-)U_j^n + (-\frac{\Delta t}{\Delta x} \Lambda^-)U_{j+1}^n.$$

With this equation, we construct a mapping $F : V \rightarrow W$ as follows

$$(I - \frac{\Delta t}{\epsilon} \mathcal{Q}(V, 0))W = \frac{\Delta t}{\Delta x} \Lambda^+ U_{j-1}^n + (I - \frac{\Delta t}{\Delta x} \Lambda^+ + \frac{\Delta t}{\Delta x} \Lambda^-)U_j^n + (-\frac{\Delta t}{\Delta x} \Lambda^-)U_{j+1}^n.$$

Due to Corollary 2.2, we can write

$$W = (I - \frac{\Delta t}{\epsilon} \mathcal{Q}(V, 0))^{-1} [\frac{\Delta t}{\Delta x} \Lambda^+ U_{j-1}^n + (I - \frac{\Delta t}{\Delta x} \Lambda^+ + \frac{\Delta t}{\Delta x} \Lambda^-)U_j^n + (-\frac{\Delta t}{\Delta x} \Lambda^-)U_{j+1}^n]$$

and thereby

$$\begin{aligned} |W| &\leq |(I - \frac{\Delta t}{\epsilon} \mathcal{Q}(V, 0))^{-1}| |\frac{\Delta t}{\Delta x} \Lambda^+ U_{j-1}^n + (I - \frac{\Delta t}{\Delta x} \Lambda^+ + \frac{\Delta t}{\Delta x} \Lambda^-)U_j^n + (-\frac{\Delta t}{\Delta x} \Lambda^-)U_{j+1}^n| \\ &= |\frac{\Delta t}{\Delta x} \Lambda^+ U_{j-1}^n + (I - \frac{\Delta t}{\Delta x} \Lambda^+ + \frac{\Delta t}{\Delta x} \Lambda^-)U_j^n + (-\frac{\Delta t}{\Delta x} \Lambda^-)U_{j+1}^n|. \end{aligned}$$

Therefore, W is bounded for fixed j and n . By the Brouwer fixed-point theorem, the mapping F has fixed points for F is continuous. This shows the existence of U_j^{n+1} .

To see the uniqueness, we assume that both U_1 and U_2 satisfy the difference scheme (2.2). With the mean-value theorem, we see that

$$U_1 - U_2 = \frac{\Delta t}{\epsilon} (Q(U_1) - Q(U_2)) = \frac{\Delta t}{\epsilon} \mathcal{Q}(U_1, U_2)(U_1 - U_2)$$

and thereby

$$(I - \frac{\Delta t}{\epsilon} \mathcal{Q}(U_1, U_2))(U_1 - U_2) = 0.$$

It follows from Corollary 2.2 that $U_1 = U_2$. This shows the uniqueness and hence the proof is complete. \square

Remark 2.1. We could use the contraction mapping principle to show the existence and uniqueness of U_j^{n+1} . However, it requires that Δt is much smaller than ϵ . On the other hand, this lemma does not tell the uniform boundedness of the difference solution U_j^n .

Next we establish the L^1 -stability of the difference scheme (2.2).

Lemma 2.4. *Let U_j^n and V_j^n be two solutions of the difference scheme (2.2) with initial data U_j^0 and V_j^0 , respectively. Then it holds that*

$$\sum_{j=-\infty}^{+\infty} |U_j^n - V_j^n| \leq \sum_{j=-\infty}^{+\infty} |U_j^0 - V_j^0|$$

for all $n \geq 0$.

Proof. It follows from the difference scheme (2.2) that

$$\begin{aligned} U_j^{n+1} - V_j^{n+1} &= \frac{\Delta t}{\epsilon} (Q(U_j^{n+1}) - Q(V_j^{n+1})) + \frac{\Delta t}{\Delta x} \Lambda^+(U_{j-1}^n - V_{j-1}^n) \\ &\quad + (I - \frac{\Delta t}{\Delta x} \Lambda^+ + \frac{\Delta t}{\Delta x} \Lambda^-)(U_j^n - V_j^n) + (-\frac{\Delta t}{\Delta x} \Lambda^-)(U_{j+1}^n - V_{j+1}^n). \end{aligned}$$

Thus we deduce from the mean-value theorem and Corollary 2.2 that

$$\begin{aligned} (I - \frac{\Delta t}{\epsilon} \mathcal{Q}(U_j^{n+1}, V_j^{n+1}))(U_j^{n+1} - V_j^{n+1}) &= \frac{\Delta t}{\Delta x} \Lambda^+(U_{j-1}^n - V_{j-1}^n) \\ &\quad + (I - \frac{\Delta t}{\Delta x} \Lambda^+ + \frac{\Delta t}{\Delta x} \Lambda^-)(U_j^n - V_j^n) + (-\frac{\Delta t}{\Delta x} \Lambda^-)(U_{j+1}^n - V_{j+1}^n), \end{aligned}$$

$$\begin{aligned} U_j^{n+1} - V_j^{n+1} &= (I - \frac{\Delta t}{\epsilon} \mathcal{Q}(U_j^{n+1}, V_j^{n+1}))^{-1} [\frac{\Delta t}{\Delta x} \Lambda^+(U_{j-1}^n - V_{j-1}^n) \\ &\quad + (I - \frac{\Delta t}{\Delta x} \Lambda^+ + \frac{\Delta t}{\Delta x} \Lambda^-)(U_j^n - V_j^n) + (-\frac{\Delta t}{\Delta x} \Lambda^-)(U_{j+1}^n - V_{j+1}^n)], \end{aligned}$$

$$\begin{aligned} |U_j^{n+1} - V_j^{n+1}| &\leq |(I - \frac{\Delta t}{\epsilon} \mathcal{Q}(U_j^{n+1}, V_j^{n+1}))^{-1}| |\frac{\Delta t}{\Delta x} \Lambda^+(U_{j-1}^n - V_{j-1}^n) \\ &\quad + (I - \frac{\Delta t}{\Delta x} \Lambda^+ + \frac{\Delta t}{\Delta x} \Lambda^-)(U_j^n - V_j^n) + (-\frac{\Delta t}{\Delta x} \Lambda^-)(U_{j+1}^n - V_{j+1}^n)| \\ &\leq |\frac{\Delta t}{\Delta x} \Lambda^+(U_{j-1}^n - V_{j-1}^n)| \\ &\quad + |(I - \frac{\Delta t}{\Delta x} \Lambda^+ + \frac{\Delta t}{\Delta x} \Lambda^-)(U_j^n - V_j^n)| + |(-\frac{\Delta t}{\Delta x} \Lambda^-)(U_{j+1}^n - V_{j+1}^n)|, \end{aligned}$$

and thereby

$$\begin{aligned}
\sum_{j=-\infty}^{+\infty} |U_j^{n+1} - V_j^{n+1}| &\leq \sum_{j=-\infty}^{+\infty} \left| \frac{\Delta t}{\Delta x} \Lambda^+(U_{j-1}^n - V_{j-1}^n) \right| \\
&\quad + \sum_{j=-\infty}^{+\infty} \left| \left(I - \frac{\Delta t}{\Delta x} \Lambda^+ + \frac{\Delta t}{\Delta x} \Lambda^- \right) (U_j^n - V_j^n) \right| + \sum_{j=-\infty}^{+\infty} \left| \left(-\frac{\Delta t}{\Delta x} \Lambda^- \right) (U_{j+1}^n - V_{j+1}^n) \right| \\
&= \sum_{j=-\infty}^{+\infty} \left[\left| \frac{\Delta t}{\Delta x} \Lambda^+(U_j^n - V_j^n) \right| + \left| \left(I - \frac{\Delta t}{\Delta x} \Lambda^+ + \frac{\Delta t}{\Delta x} \Lambda^- \right) (U_j^n - V_j^n) \right| \right. \\
&\quad \left. + \left| \left(-\frac{\Delta t}{\Delta x} \Lambda^- \right) (U_j^n - V_j^n) \right| \right].
\end{aligned}$$

In the last step we use the induction assumption that $\sum_{j=-\infty}^{+\infty} |U_j^n - V_j^n| < \infty$, which is true if $\sum_{j=-\infty}^{+\infty} |U_j^0 - V_j^0| < \infty$. On the other hand, from the definition of λ_i^\pm (2.3) and the CFL-condition (2.5) we deduce that

$$\begin{aligned}
&\left| \frac{\Delta t}{\Delta x} \Lambda^+(U_j^n - V_j^n) \right| + \left| \left(I - \frac{\Delta t}{\Delta x} \Lambda^+ + \frac{\Delta t}{\Delta x} \Lambda^- \right) (U_j^n - V_j^n) \right| + \left| \left(-\frac{\Delta t}{\Delta x} \Lambda^- \right) (U_j^n - V_j^n) \right| \\
&= \sum_{i=1}^r \frac{\Delta t}{\Delta x} \lambda_i^+ |u_{ij}^n - v_{ij}^n| + \sum_{i=1}^r \left(1 - \frac{\Delta t}{\Delta x} \lambda_i^+ + \frac{\Delta t}{\Delta x} \lambda_i^- \right) |u_{ij}^n - v_{ij}^n| - \sum_{i=1}^r \frac{\Delta t}{\Delta x} \lambda_i^- |u_{ij}^n - v_{ij}^n| \\
&= |U_j^n - V_j^n|.
\end{aligned}$$

Hence we see that

$$\sum_{j=-\infty}^{+\infty} |U_j^{n+1} - V_j^{n+1}| \leq \sum_{j=-\infty}^{+\infty} |U_j^n - V_j^n| \leq \dots \leq \sum_{j=-\infty}^{+\infty} |U_j^0 - V_j^0|.$$

This completes the proof. \square

By taking $V_j^n = U_{j-1}^n$ in Lemma 2.4, we get the following corollary on BV-estimates of the difference solutions.

Corollary 2.5. *Let U_j^n be a solution to the difference scheme (2.2) with initial data U_j^0 . Then the BV-estimate*

$$\sum_{j=-\infty}^{+\infty} |U_j^n - U_{j-1}^n| \leq \sum_{j=-\infty}^{+\infty} |U_j^0 - U_{j-1}^0|$$

holds for all $n \geq 0$.

Having Corollary 2.5, we show the uniform boundedness of the difference solutions, which is not covered in Lemma 2.3.

Lemma 2.6.

$$\sup_j |U_j^n| \leq \sum_{j=-\infty}^{+\infty} |U_j^0 - U_{j-1}^0|$$

holds for all $n \geq 0$.

Proof. Thanks to the initial data (2.4), we have $U_{j_0}^n = 0$ if j_0 is large enough. Thus, for any j it follows that

$$|U_j^n| = |U_j^n - U_{j_0}^n| \leq \sum_{k=j+1}^{j_0} |U_k^n - U_{k-1}^n| \leq \sum_{k=-\infty}^{+\infty} |U_k^0 - U_{k-1}^0|.$$

This completes the proof. \square

Remark 2.2. This lemma requires that the initial data are of bounded variation. Otherwise, we could not obtain the uniform boundedness of the difference solutions.

3. TIME-LIPSCHITZ CONTINUITY

In this section we show the time-Lipschitz continuity of the difference solutions, which seems not so easy for the present problem.

To begin with, we set $G(U) = (f_1, f_2, \dots, f_{r-1})^T \in \mathbb{R}^{r-1}$ and $G_j^n = G(U_j^n)$. Note that

$$(3.1) \quad Q(U) = KG(U), \quad G(U) = K'Q(U),$$

where K is the constant $r \times (r-1)$ -matrix

$$(3.2) \quad K = \begin{pmatrix} 1 & & & & \\ -1 & 1 & & & \\ & -1 & 1 & & \\ & & \cdots & \cdots & \cdots \\ & & & -1 & 1 \\ & & & & -1 \end{pmatrix}.$$

and K' is the constant $(r-1) \times r$ -matrix such that $K'K = I_{r-1}$. Moreover, it follows from the mean-value theorem and the definitions in (2.6) and (2.7) that

$$G_j^{n+1} - G_j^n = \mathcal{M}(U_j^n, U_j^{n+1})(U_j^{n+1} - U_j^n)$$

with $\mathcal{M}(U_j^n, U_j^{n+1})$ the following $(r-1) \times r$ matrix

$$\mathcal{M}(U_j^n, U_j^{n+1}) = \begin{pmatrix} A_1 & B_1 & & & \\ & A_2 & B_2 & & \\ & & A_3 & B_3 & \\ & & \cdots & \cdots & \cdots \\ & & & A_{r-1} & B_{r-1} \end{pmatrix}.$$

Set $M = M(U_j^n, U_j^{n+1}) = \mathcal{M}(U_j^n, U_j^{n+1})K \in \mathbb{R}^{(r-1) \times (r-1)}$. By calculation, we have

$$M = \begin{pmatrix} A_1 - B_1 & B_1 & & & \\ -A_2 & A_2 - B_2 & B_2 & & \\ & -A_3 & A_3 - B_3 & B_3 & \\ & & \cdots & \cdots & \cdots \\ & & & -A_{r-1} & A_{r-1} - B_{r-1} \end{pmatrix}.$$

Thanks to the strict monotonicity assumption (1) on the f_i 's, it is an elementary fact that $I - \frac{\Delta t}{\epsilon} M$ is invertible.

We assume that there exists a (U_j^n, U_j^{n+1}) -independent norm $|\cdot|_*$ on \mathbf{R}^{r-1} and a positive constant λ such that

$$(3.3) \quad |(I - \frac{\Delta t}{\epsilon} M)^{-1}|_* \leq (1 + \frac{\Delta t}{\epsilon} \lambda)^{-1}.$$

Although such a norm has not been found for general cases with $r > 3$, this assumption is indeed true for $r \leq 3$ and for the case where $Q(U)$ is linear with respect to U . In fact, we have

Proposition 3.1. *For $r \leq 3$, there exists a positive constant λ such that*

$$|(I - \frac{\Delta t}{\epsilon} M)^{-1}|_\infty \leq (1 + \frac{\Delta t}{\epsilon} \lambda)^{-1}.$$

Proof. When $r = 2$, M is the negative number $(A_1 - B_1)$. Thus we have

$$|(I - \frac{\Delta t}{\epsilon} M)^{-1}|_\infty = (1 + \frac{\Delta t}{\epsilon} (B_1 - A_1))^{-1} \leq (1 + \frac{\Delta t}{\epsilon} \lambda)^{-1}$$

with $\lambda = \min_{V,W} \{B_1(V, W) - A_1(V, W)\} > 0$. Here the boundedness of the different solution established in Lemma 2.6 has been used.

For $r = 3$, M is the 2×2 matrix

$$M = \begin{pmatrix} A_1 - B_1 & B_1 \\ -A_2 & A_2 - B_2 \end{pmatrix}.$$

Then we have

$$I - \frac{\Delta t}{\epsilon} M = \begin{pmatrix} 1 + \frac{\Delta t}{\epsilon} (B_1 - A_1) & -\frac{\Delta t}{\epsilon} B_1 \\ \frac{\Delta t}{\epsilon} A_2 & 1 + \frac{\Delta t}{\epsilon} (B_2 - A_2) \end{pmatrix}$$

and

$$\begin{aligned} & (I - \frac{\Delta t}{\epsilon} M)^{-1} \\ &= \frac{1}{1 + A \frac{\Delta t}{\epsilon} + B \frac{\Delta t^2}{\epsilon^2}} \begin{pmatrix} 1 + \frac{\Delta t}{\epsilon} (B_2 - A_2) & \frac{\Delta t}{\epsilon} B_1 \\ -\frac{\Delta t}{\epsilon} A_2 & 1 + \frac{\Delta t}{\epsilon} (B_1 - A_1) \end{pmatrix} \end{aligned}$$

with $A = B_1 - A_1 + B_2 - A_2$ and $B = B_1 B_2 - A_1 B_2 + A_1 A_2$. Recall that $A_i < 0$ and $B_i > 0$ for $i = 1, 2$. Set

$$a = \max\{B_1 + B_2 - A_2, B_1 - A_1 - A_2\}.$$

It is obvious that $0 < a < A$ and $B > 0$. Thus we deduce that

$$|(I - \frac{\Delta t}{\epsilon} M)^{-1}|_\infty = \frac{1 + a \frac{\Delta t}{\epsilon}}{1 + A \frac{\Delta t}{\epsilon} + B \frac{\Delta t^2}{\epsilon^2}} \leq (1 + \frac{\Delta t}{\epsilon} \lambda)^{-1}$$

with

$$\lambda = \min\{A - a, B/a\} > 0.$$

This completes the proof. \square

For $r > 3$, M is no longer strictly diagonally dominant and the inequality in Proposition 3.1 does not hold anymore. However, we have

Proposition 3.2. *If $Q(U)$ is linear with respect to U , then the above assumption (3.3) holds.*

Proof. Since $Q(U)$ is linear with respect to U , M is a constant matrix. Set $\alpha_1 = 1, \alpha_{i+1} = -\alpha_i \frac{B_i}{A_{i+1}}$, and $D = \text{diag}(\alpha_1, \alpha_2, \dots, \alpha_{r-1})$. It is obvious that the diagonal matrix D is positive definite and DM is symmetric. Then $D^{\frac{1}{2}}MD^{-\frac{1}{2}}$ is also symmetric. Thus, there exist an orthogonal matrix T and a diagonal negative-definite matrix N such that

$$D^{\frac{1}{2}}MD^{-\frac{1}{2}} = T^{-1}NT,$$

and thereby

$$(TD^{\frac{1}{2}})(I - \frac{\Delta t}{\epsilon}M)^{-1} = (I - \frac{\Delta t}{\epsilon}N)^{-1}(TD^{\frac{1}{2}}).$$

Set $P = TD^{\frac{1}{2}}$. We define a norm on \mathbf{R}^{r-1} as

$$|\xi|_p = |P\xi|$$

for $\xi \in \mathbf{R}^{r-1}$. Thus the corresponding matrix norm is

$$\begin{aligned} |(I - \frac{\Delta t}{\epsilon}M)^{-1}|_{\star} &:= \sup_{|\xi|_p \leq 1} |(I - \frac{\Delta t}{\epsilon}M)^{-1}\xi|_p \\ &= \sup_{|\xi|_p \leq 1} |P(I - \frac{\Delta t}{\epsilon}M)^{-1}\xi| \\ &= \sup_{|\xi|_p \leq 1} |(I - \frac{\Delta t}{\epsilon}N)^{-1}P\xi| \\ &\leq \sup_{|\xi|_p \leq 1} |(I - \frac{\Delta t}{\epsilon}N)^{-1}||P\xi| \\ &\leq |(I - \frac{\Delta t}{\epsilon}N)^{-1}| = (1 + \frac{\Delta t}{\epsilon}\lambda)^{-1}, \end{aligned}$$

where λ is the smallest eigenvalue of $-N$. □

Now we turn to estimate $Q^n = Q(U_j^n)$.

Lemma 3.3. *Assume (3.3) holds. Then there is a positive constant C such that*

$$\|Q^n\|_{L^1} \leq C((1 + \lambda \frac{\Delta t}{\epsilon})^{-n} \|Q^0\|_{L^1} + \epsilon).$$

for all n .

Proof. From the difference scheme it follows that

$$\begin{aligned} G_j^{n+1} - G_j^n &= \mathcal{M}(U_j^n, U_j^{n+1})(U_j^{n+1} - U_j^n) \\ &= \mathcal{M}(U_j^n, U_j^{n+1})[\frac{\Delta t}{\epsilon}Q_j^{n+1} - \frac{\Delta t}{\Delta x}\Lambda^+(U_j^n - U_{j-1}^n) - \frac{\Delta t}{\Delta x}\Lambda^-(U_{j+1}^n - U_j^n)] \\ &= \frac{\Delta t}{\epsilon}MG_j^{n+1} - \mathcal{M}(U_j^n, U_j^{n+1})[\Lambda^+(U_j^n - U_{j-1}^n) + \Lambda^-(U_{j+1}^n - U_j^n)]\frac{\Delta t}{\Delta x} \end{aligned}$$

and thereby

$$\begin{aligned} (I - \frac{\Delta t}{\epsilon}M)G_j^{n+1} &= G_j^n - \mathcal{M}(U_j^n, U_j^{n+1})[\Lambda^+(U_j^n - U_{j-1}^n) \\ &\quad + \Lambda^-(U_{j+1}^n - U_j^n)]\frac{\Delta t}{\Delta x}. \end{aligned}$$

Thus we deduce from the inequality (3.3) that

$$\begin{aligned}
(1 + \lambda \frac{\Delta t}{\epsilon}) |G_j^{n+1}|_* &\leq |(I - \frac{\Delta t}{\epsilon} M) G_j^{n+1}|_* \\
&= |G_j^n - \mathcal{M}(U_j^n, U_j^{n+1}) [\Lambda^+(U_j^n - U_{j-1}^n) + \Lambda^-(U_{j+1}^n - U_j^n)] \frac{\Delta t}{\Delta x}|_* \\
&\leq |G_j^n|_* + |\mathcal{M}(U_j^n, U_j^{n+1}) [\Lambda^+(U_j^n - U_{j-1}^n) + \Lambda^-(U_{j+1}^n - U_j^n)]|_* \frac{\Delta t}{\Delta x}.
\end{aligned}$$

Since $|\cdot|_*$ is equivalent to the 1-norm $|\cdot|$ on \mathbf{R}^{r-1} and since the difference solution is bounded, there is a positive constant C such that

$$(1 + \lambda \frac{\Delta t}{\epsilon}) |G_j^{n+1}|_* \leq |G_j^n|_* + C \frac{\Delta t}{\Delta x} (|U_j^n - U_{j-1}^n| + |U_{j+1}^n - U_j^n|).$$

Consequently, we use Corollary 2.5 to obtain

$$\begin{aligned}
(1 + \lambda \frac{\Delta t}{\epsilon}) \sum_j |G_j^{n+1}|_* \Delta x &\leq \sum_j |G_j^n|_* \Delta x + C \Delta t \sum_j |U_j^n - U_{j-1}^n| \\
&\leq \sum_j |G_j^n|_* \Delta x + C \Delta t.
\end{aligned}$$

From the last inequality, it is easy to verify that

$$\sum_j |G_j^n|_* \Delta x \leq (1 + \lambda \frac{\Delta t}{\epsilon})^{-n} \sum_j |G_j^0|_* \Delta x + C \epsilon \leq C ((1 + \lambda \frac{\Delta t}{\epsilon})^{-n} \sum_j |G_j^0|_* \Delta x + \epsilon).$$

Thus we get

$$\|G^n\|_{L^1} \leq C \sum_j |G_j^n|_* \Delta x \leq C ((1 + \lambda \frac{\Delta t}{\epsilon})^{-n} \sum_j |G_j^0|_* \Delta x + \epsilon).$$

Finally, from (3.1) it follows that

$$\|Q^n\|_{L^1} \leq C \|G^n\|_{L^1} \quad \text{and} \quad \|G^0\|_{L^1} \leq C \|Q^0\|_{L^1}.$$

Hence

$$\|Q^n\|_{L^1} \leq C ((1 + \lambda \frac{\Delta t}{\epsilon})^{-n} \|Q^0\|_{L^1} + \epsilon)$$

and the proof is complete. \square

Now we can easily show the time-Lipschitz continuity of U^n .

Lemma 3.4. *Assume (3.3) holds. Then*

$$\|U_j^{n+1} - U_j^n\|_{L^1} \leq C (\frac{\Delta t}{\epsilon} (1 + \lambda \frac{\Delta t}{\epsilon})^{-(n+1)} \|Q^0\|_{L^1} + \Delta t)$$

for $n = 0, 1, 2, \dots$.

Proof. From the difference scheme (2.2) we have

$$U_j^{n+1} - U_j^n = \frac{\Delta t}{\epsilon} Q_j^{n+1} - \frac{\Delta t}{\Delta x} \Lambda^+(U_j^n - U_{j-1}^n) - \frac{\Delta t}{\Delta x} \Lambda^-(U_{j+1}^n - U_j^n)$$

and therefore

$$|U_j^{n+1} - U_j^n| \leq \frac{\Delta t}{\epsilon} |Q_j^{n+1}| + C \frac{\Delta t}{\Delta x} (|U_j^n - U_{j-1}^n| + |U_{j+1}^n - U_j^n|).$$

Thus, from Corollary 2.5 and Lemma 3.3 we deduce that

$$\|U_j^{n+1} - U_j^n\|_{L^1} \leq \frac{\Delta t}{\epsilon} \|Q_j^{n+1}\|_{L^1} + C \Delta t \leq C \left(\frac{\Delta t}{\epsilon} (1 + \lambda \frac{\Delta t}{\epsilon})^{-(n+1)} \|Q^0\|_{L^1} + \Delta t \right).$$

This completes the proof. \square

4. ENTROPIES

In this section, we discuss entropy functions for the reaction-hyperbolic system (1.1) and its equilibrium system (1.2). For this purpose, we make the following additional assumption

(3). For any fixed v , there exists w such that $vf_i(v, w) \geq 0$ for $i = 1, 2, \dots, r-1$.

This assumption, together with those made in the previous section, ensures that there exists a unique and globally-defined function h_i of one variable such that

$$(4.1) \quad f_i(u_i, u_{i+1}) = 0 \quad \text{iff} \quad u_{i+1} = h_i(u_i)$$

for $i = 1, 2, \dots, r-1$. Obviously, h_i is strictly increasing and $h_i(0) = 0$. By the implicit function theorem, h_i is continuously differentiable. Thus the equilibrium system (1.2) can be written as

$$(4.2) \quad \begin{aligned} \partial_t(u_1 + u_2 + \dots + u_r) + \partial_x(\lambda_1 u_1 + \lambda_2 u_2 + \dots + \lambda_r u_r) &= 0, \\ u_2 &= h_1(u_1), \quad u_3 = h_2(u_2), \quad \dots, \quad u_r = h_{r-1}(u_{r-1}). \end{aligned}$$

Set

$$v = u_1 + h_1(u_1) + h_2 \circ h_1(u_1) + \dots + h_{r-1} \circ h_{r-2} \circ \dots \circ h_1(u_1).$$

Since the right-hand side is strictly increasing with respect to u_1 , u_1 can be expressed as a function of v , say $u_1 = u_1(v)$. By the inverse function theorem, $u_1(v)$ is continuously differentiable. Set

$$u_{i+1}(v) = h_i(u_i(v))$$

for $i = 1, 2, \dots, r-1$. Consequently, each $u_i(v)$ is strictly increasing, continuously differentiable and $u_i(0) = 0$. Set

$$v = u_1(v) + u_2(v) + \dots + u_r(v) \quad \text{and} \quad h(v) = \lambda_1 u_1(v) + \lambda_2 u_2(v) + \dots + \lambda_r u_r(v).$$

Then the equilibrium system (4.2) can be rewritten as

$$(4.3) \quad \partial_t v + \partial_x h(v) = 0.$$

Recall that any convex function is a convex entropy function for scalar conservation laws like (4.3) (see, e.g., [3]).

Next, we turn to discuss the entropy functions for system (4.3) and the reaction-hyperbolic system (1.1).

Lemma 4.1. *Given a strictly convex smooth function $\tilde{\eta}(v)$, there is a dissipative entropy function $\eta(U)$, in the sense of [11], for the reaction-hyperbolic system (1.1) such that*

$$\tilde{\eta}(v) = \eta(u_1(v), u_2(v), \dots, u_r(v)) \equiv \eta(U(v)).$$

Proof. We inductively define

$$\begin{aligned} \eta_r(u) &= \int_0^u \tilde{\eta}'(u_r^{-1}(w)) dw, \\ \eta_{i-1}(u) &= \int_0^u \eta'_i(h_{i-1}(w)) dw \end{aligned}$$

for $i = r, r-1, \dots, 2$. Then we have

$$(4.4) \quad \eta'_{i-1}(u_{i-1}) = \eta'_i(h_{i-1}(u_{i-1})) \quad \text{and} \quad \eta'_r(u) = \tilde{\eta}'(u_r^{-1}(u)).$$

Since $\tilde{\eta}(v)$ is strictly convex and u_r^{-1} and h_i are all strictly increasing, it is clear that $\eta'_1, \eta'_2, \dots, \eta'_r$ are strictly increasing. Therefore,

$$(4.5) \quad \eta(U) := \sum_{i=1}^r \eta_i(u_i) + \tilde{\eta}(0)$$

is a strictly convex function of U . Recall the matrix K defined in (3.2). We see that

$$(\eta'_1 - \eta'_2, \eta'_2 - \eta'_3, \dots, \eta'_{r-1} - \eta'_r)^T = K^T \eta_U(U)$$

and, moreover, from (3.1) that

$$\begin{aligned} (4.6) \quad Q(U) &= KG(U) \\ &= K \operatorname{diag}\left(\frac{f_1(u_1, u_2)}{\eta'_1 - \eta'_2}, \frac{f_2(u_2, u_3)}{\eta'_2 - \eta'_3}, \dots, \frac{f_{r-1}(u_{r-1}, u_r)}{\eta'_{r-1} - \eta'_r}\right) K^T \eta_U(U) \\ &\equiv S(U) \eta_U(U). \end{aligned}$$

Thanks to the relations in (4.4), we deduce from (4.1), the convexity of η_i and the monotonicity assumption (1) that

$$\begin{aligned} \frac{f_i(u_i, u_{i+1})}{\eta'_i - \eta'_{i+1}} &= \frac{f_i(u_i, u_{i+1})}{\eta'_{i+1}(h_i(u_i)) - \eta'_{i+1}(u_{i+1})} \\ &= - \frac{\int_0^1 f_{iu_{i+1}}(u_i, h_i(u_i) + \sigma(u_{i+1} - h_i(u_i))) d\sigma}{\int_0^1 \eta''_{i+1}(h_i(u_i) + \sigma(u_{i+1} - h_i(u_i))) d\sigma} < 0. \end{aligned}$$

Thus, $S(U)$ is a symmetric and non-positive definite matrix. Its null space is obviously that of K^T , which is independent of U . Consequently, $\eta(U)$ is a dissipative entropy function, in the sense of [11], for the reaction-hyperbolic system (1.1).

Furthermore, we deduce from the relations in (4.4) that

$$\begin{aligned} \eta'_1(u_1(v)) &= \eta'_2(h_1(u_1(v))) \\ &= \eta'_3(h_2 \circ h_1(u_1(v))) \\ &= \dots \\ &= \eta'_r(h_{r-1} \circ h_{r-2} \circ \dots \circ h_1(u_1(v))) \\ &= \eta'_r(u_r(v)) \\ &= \tilde{\eta}'(v). \end{aligned}$$

Similarly, we have

$$\eta'_2(u_2(v)) = \cdots = \eta'_r(u_r(v)) = \tilde{\eta}'(v).$$

Therefore, we deduce from $u_1(v) + u_2(v) + \cdots + u_r(v) \equiv v$ that

$$\begin{aligned} (\eta(U(v)))' &= \left(\sum_{i=1}^r \eta_i(u_i(v)) \right)' \\ &= \eta'_1(u_1(v))u'_1(v) + \eta'_2(u_2(v))u'_2(v) + \cdots + \eta'_r(u_r(v))u'_r(v) \\ &= \tilde{\eta}'(v). \end{aligned}$$

In view of $\eta_i(0) = 0 = u_i(0)$, we see from (4.5) that $\eta(U(0)) = \tilde{\eta}(0)$ and hence

$$\tilde{\eta}(v) = \eta(U(v)).$$

This completes the proof. \square

We conclude this section with a discrete entropy inequality for the difference solutions.

Lemma 4.2. *Let U_j^n be a solution to the difference scheme (2.2). Then, for any smooth convex function $\eta(U) = \sum_{i=1}^r \eta_i(u_i)$, there exists a Lipschitz continuous function Ψ of two variables such that for all $j \in \mathbb{Z}$ and $n \geq 0$, the following cell entropy inequalities hold:*

$$\eta(U_j^{n+1}) \leq \eta(U_j^n) - \frac{\Delta t}{\Delta x} (\Psi(U_j^n, U_{j+1}^n) - \Psi(U_{j-1}^n, U_j^n)) + \frac{\Delta t}{\epsilon} \eta_U(U_j^{n+1}) Q(U_j^{n+1}).$$

Moreover, the Lipschitz continuous function satisfies the consistency relation

$$\Psi(U, U) = \sum_{i=1}^r \lambda_i \eta_i(u_i).$$

Proof. For any smooth convex function η_i and any two real numbers $a, b \in \mathbb{R}$, it is standard that

$$\eta_i(b) - \eta_i(a) \leq \eta'_i(b)(b - a).$$

Thus, for the given convex function $\eta(U) = \sum_{i=1}^r \eta_i(u_i)$ and for any $V, W \in \mathbb{R}^r$ we have

$$\eta(V) - \eta(W) \leq \eta_U(V)(V - W).$$

Thus, it follows from the original difference scheme that

$$\eta(U_j^{n+1}) \leq \eta(U_j^n - \frac{\Delta t}{\Delta x} \Lambda^+(U_j^n - U_{j-1}^n) - \frac{\Delta t}{\Delta x} \Lambda^-(U_{j+1}^n - U_j^n)) + \eta_U(U_j^{n+1}) \frac{\Delta t}{\epsilon} Q_j^{n+1},$$

since

$$U_j^{n+1} = \frac{\Delta t}{\epsilon} Q_j^{n+1} + U_j^n - \frac{\Delta t}{\Delta x} \Lambda^+(U_j^n - U_{j-1}^n) - \frac{\Delta t}{\Delta x} \Lambda^-(U_{j+1}^n - U_j^n).$$

On the other hand, we deduce from (2.3), (2.5) and the convexity of η_i that

$$\begin{aligned}
& \eta(U_j^n - \frac{\Delta t}{\Delta x} \Lambda^+(U_j^n - U_{j-1}^n) - \frac{\Delta t}{\Delta x} \Lambda^-(U_{j+1}^n - U_j^n)) \\
&= \eta(\frac{\Delta t}{\Delta x} \Lambda^+ U_{j-1}^n + (I - \frac{\Delta t}{\Delta x} \Lambda^+ + \frac{\Delta t}{\Delta x} \Lambda^-) U_j^n - \frac{\Delta t}{\Delta x} \Lambda^- U_{j+1}^n) \\
&= \sum_{i=1}^r \eta_i(\lambda_i^+ \frac{\Delta t}{\Delta x} u_{i,j-1}^n + (1 - \lambda_i^+ \frac{\Delta t}{\Delta x} + \lambda_i^- \frac{\Delta t}{\Delta x}) u_{i,j}^n - \lambda_i^- \frac{\Delta t}{\Delta x} u_{i,j+1}^n) \\
&\leq \sum_{i=1}^r [\lambda_i^+ \frac{\Delta t}{\Delta x} \eta_i(u_{i,j-1}^n) + (1 - \lambda_i^+ \frac{\Delta t}{\Delta x} + \lambda_i^- \frac{\Delta t}{\Delta x}) \eta_i(u_{i,j}^n) - \lambda_i^- \frac{\Delta t}{\Delta x} \eta_i(u_{i,j+1}^n)] \\
&= \sum_{i=1}^r [\eta_i(u_{i,j}^n) - \frac{\Delta t}{\Delta x} (|\lambda_i| \eta_i(u_{i,j}^n) - \lambda_i^+ \eta_i(u_{i,j-1}^n) + \lambda_i^- \eta_i(u_{i,j+1}^n))].
\end{aligned}$$

Define

$$\Psi(U, V) = \sum_{i=1}^r [\frac{\lambda_i}{2} (\eta_i(u_i) + \eta_i(v_i)) + \frac{|\lambda_i|}{2} (\eta_i(u_i) - \eta_i(v_i))].$$

This Ψ is obviously Lipschitz continuous and satisfies the consistency relation. Moreover, the above inequalities lead directly to

$$\eta(U_j^{n+1}) \leq \eta(U_j^n) - \frac{\Delta t}{\Delta x} (\Psi(U_j^n, U_{j+1}^n) - \Psi(U_{j-1}^n, U_j^n)) + \frac{\Delta t}{\epsilon} \eta_U(U_j^{n+1}) Q(U_j^{n+1}).$$

This completes the proof. \square

5. MAIN RESULTS

In this section we prove the main results of this paper. To this end, we define

$$U^\Delta(x, t) = (u_1^\Delta(x, t), u_2^\Delta(x, t), \dots, u_r^\Delta(x, t))^T := (u_{1,j}^n, u_{2,j}^n, \dots, u_{r,j}^n)^T$$

for $(x, t) \in [j\Delta x, (j+1)\Delta x) \times [n\Delta t, (n+1)\Delta t)$. With this definition, it simply follows from Corollary 2.5, Lemma 2.6 and Lemma 3.4 that

Lemma 5.1. *The piecewise constant function $U^\Delta(x, t)$ satisfies the following estimates*

$$(5.1) \quad |U^\Delta(x, t)| \leq \sum_{j=-\infty}^{+\infty} |U_j^0 - U_{j-1}^0| \quad \text{for all } (x, t),$$

$$(5.2) \quad TV(U^\Delta(\cdot, t)) \leq \sum_{j=-\infty}^{+\infty} |U_j^0 - U_{j-1}^0|,$$

$$(5.3) \quad \|U^\Delta(\cdot, t) - U^\Delta(\cdot, t_1)\|_{L^1} \leq C(\frac{1}{\epsilon}(1 + \lambda \frac{\Delta t}{\epsilon})^{-\frac{\min\{t, t_1\}}{\Delta t}} \|Q^0\|_{L^1} + 1)(|t - t_1| + \Delta t)$$

for all $t, t_1 > 0$.

Proof. We only need to show the last inequality. Let k, k_1 be two integers such that $t \in [k\Delta t, (k+1)\Delta t), t_1 \in [k_1\Delta t, (k_1+1)\Delta t)$. Without loss of generality, we assume $k_1 \leq k$. Then we deduce from the definition of U^Δ and Lemma 3.4 that

$$\begin{aligned}
\|U^\Delta(\cdot, t) - U^\Delta(\cdot, t_1)\|_{L^1} &= \sum_j |U_j^k - U_j^{k_1}| \Delta x \\
&\leq \sum_{n=k_1}^{k-1} \sum_j |U_j^{n+1} - U_j^n| \Delta x \\
&\leq \sum_{n=k_1}^{k-1} C \left(\frac{1}{\epsilon} \left(1 + \lambda \frac{\Delta t}{\epsilon} \right)^{-(n+1)} \|Q^0\|_{L^1} + 1 \right) \Delta t \\
&\leq C \left(\frac{1}{\epsilon} \left(1 + \lambda \frac{\Delta t}{\epsilon} \right)^{-(k_1+1)} \|Q^0\|_{L^1} + 1 \right) (k - k_1) \Delta t \\
&\leq C \left(\frac{1}{\epsilon} \left(1 + \lambda \frac{\Delta t}{\epsilon} \right)^{-\frac{\min\{t, t_1\}}{\Delta t}} \|Q^0\|_{L^1} + 1 \right) (|t - t_1| + \Delta t).
\end{aligned}$$

This completes the proof. \square

Having the estimates in Lemma 5.1 and the discrete entropy inequality in Lemma 4.2, we follow the standard argument in [3] to obtain (ϵ is fixed)

Theorem 5.2. *Suppose the initial data $U_0(x) = (u_{10}(x), u_{20}(x), \dots, u_{r0}(x))$ have bounded variations, the grid sizes Δt and Δx satisfy the CFL-condition (2.5), the f'_i 's satisfy the assumptions (1)–(3), and the inequality (3.3) holds. Then, as the grid sizes $\Delta t, \Delta x$ tend to zero, there is a subsequence of the function family $U^\Delta(x, t) = (u_1^\Delta(x, t), u_2^\Delta(x, t), \dots, u_r^\Delta(x, t))$ converging in $(L^1_{loc}(\mathbb{R}^1 \times \mathbb{R}^+))^r$ to an entropy solution $U^\epsilon(x, t) = (u_1^\epsilon, u_2^\epsilon, \dots, u_r^\epsilon)$ of the reaction-hyperbolic system (1.1) with initial data $U_0(x)$. Furthermore, the solution fulfills the following estimates*

$$(5.4) \quad |U^\epsilon(x, t)| \leq \sum_{j=-\infty}^{+\infty} |U_j^0 - U_{j-1}^0| \quad \text{for almost all } (x, t),$$

$$(5.5) \quad TV(U^\epsilon(\cdot, t)) \leq \sum_{j=-\infty}^{+\infty} |U_j^0 - U_{j-1}^0|,$$

$$(5.6) \quad \|U^\epsilon(\cdot, t) - U^\epsilon(\cdot, t_1)\|_{L^1} \leq C \left(\frac{1}{\epsilon} \exp\left(-\frac{\lambda \min\{t, t_1\}}{\epsilon}\right) \|Q^0\|_{L^1} + 1 \right) |t - t_1|.$$

for all $t, t_1 > 0$.

In the framework of BV-solutions, the zero-relaxation limit can be very easily discussed. In fact, the standard argument in [3] proves that the embedding of $L^\infty(\mathbb{R}_+ \times \mathbb{R}) \cap L^\infty(\mathbb{R}_+, BV(\mathbb{R})) \cap Lip(\mathbb{R}_+, L^1(\mathbb{R}))$ into $L^1_{loc}(\mathbb{R}_+ \times \mathbb{R})$ is compact. On the other hand, the estimates in (5.4) – (5.6) show that $\{U^\epsilon\}_{\epsilon>0}$ lies in a bounded subset of $L^\infty(\mathbb{R}_+ \times \mathbb{R}) \cap L^\infty(\mathbb{R}_+, BV(\mathbb{R})) \cap Lip(\mathbb{R}_+, L^1(\mathbb{R}))$ by assuming

$$(5.7) \quad \|Q^0\|_{L^1} = 0.$$

Namely, the initial data are assumed to be in equilibrium. Thus, we have

Theorem 5.3. *Under the conditions of Theorem 5.2 and the equilibrium assumption (5.7), there exist a bounded measurable function $U^*(x, t) = (u_1^*(x, t), u_2^*(x, t), \dots, u_r^*(x, t))$ and a subsequence (denoted in the same way) of set $\{U^\epsilon(x, t) = (u_1^\epsilon, u_2^\epsilon, \dots, u_r^\epsilon)\}$ such that as $\epsilon \rightarrow 0$,*

$$U^\epsilon(x, t) \rightarrow U^*(x, t) \quad \text{in} \quad (L_{loc}^1(\mathbb{R}^1 \times \mathbb{R}^+))^r.$$

Moreover, the function $U^(x, t)$ is a weak entropy solution to the Cauchy problem (1.2) with initial data $U_0(x)$ and satisfies*

$$\begin{aligned} |U^*(x, t)| &\leq \sum_{j=-\infty}^{+\infty} |U_j^0 - U_{j-1}^0| \quad \text{for almost all } (x, t), \\ TV(U^*(\cdot, t)) &\leq \sum_{j=-\infty}^{+\infty} |U_j^0 - U_{j-1}^0|, \\ \|U^*(\cdot, t) - U^*(\cdot, t_1)\|_{L^1} &\leq C|t - t_1|. \end{aligned}$$

Remark 5.1. The fact that $U^*(x, t)$ satisfies the entropy conditions for the equilibrium system (1.2) follows from Lemmas 4.1 and 4.2. Indeed, because $S(U)$ in (4.6) is symmetric and non-positive, the term $\eta_U(U^{n+1})Q(U^{n+1})$ in Lemma 4.2 is

$$\eta_U(U^{n+1})Q(U^{n+1}) = \eta_U(U^{n+1})S(U^{n+1})\eta_U(U^{n+1}) \leq 0$$

if η is chosen to be that constructed in Lemma 4.1.

Remark 5.2. Without the equilibrium assumption in (5.7), U^ϵ can only converge to U^* for $t > 0$ but not up to $t = 0$, because

$$\exp(-\lambda \min\{t, t_1\}/\epsilon) \leq \frac{\epsilon}{e\lambda \min\{t, t_1\}}$$

in (5.6). Indeed, without the equilibrium assumption, initial boundary-layers occur. See also [9].

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